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Note

Numerical Solution of the Vlasov Equation by Transform Methods

Numerical techniques represent a powerful means for the investigation of nonlinear effects in plasmas, and especially for the solution of the nonlinear Vlasov equation. In this latter case, one has to deal essentially with the set of equations (written in dimensionless form):

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E(x, t) \frac{\partial f}{\partial v} = 0, \tag{1}$$

$$\frac{\partial E}{\partial x} = \int f \, dv - 1. \tag{2}$$

(The symbols have their conventional meaning and we are restricting ourselves to the one-dimensional case).

It is sometimes advantageous to transform these equations from the representation in $x - v$ space. The use of transform methods for the numerical solution of the set of Eqs. (1) and (2) has been reviewed by Armstrong *et al.* [1] and Joyce *et al.* [2] who showed that such numerical solutions are hampered by a recurrence effect. The purpose of the present note is to discuss a method whereby a pseudo collision operator is formally added to Eq. (1) in order to eliminate the recurrence effect when numerically solving the Vlasov equation via a Hermite polynomials expansion. This is done with the intent of developing a two-dimensional scheme for the numerical solution of the Vlasov equation.

The expansion of the distribution function in velocity space in terms of Hermite polynomials has been studied in particular by Grant and Feix [3], Armstrong [4], and Knorr [5]. In this case, the distribution function is expanded as follows:

$$f(x, v, t) = \sum_{\nu=0}^{\infty} b_\nu(x, t) \, H_{e\nu}(v) \, e^{-(1/2)v^2}. \tag{3}$$

When the series (3) is inserted into the Vlasov equation, and the coefficients for each Hermite polynomial are collected, one obtains the following infinite system of differential equations:

$$\frac{\partial b_\nu}{\partial t} + \frac{\partial b_{\nu-1}}{\partial x} + (\nu + 1) \frac{\partial b_{\nu+1}}{\partial x} - E(x, t) \, b_{\nu-1} = 0. \tag{4}$$

Since computers handle only finite systems, one is obliged to truncate the system in Eq. (4) at, say, $\nu = n$. This results in what appears to be a kind of numerical instability, which is in fact a recurrence effect due to the attempt to represent the continuous eigenspectrum of the infinite system, in Eq. (4), by a discrete finite spectrum of the truncated set [2].

In order to circumvent the difficulty, it was suggested that the addition of a small imaginary part to the eigenfrequencies of the truncated system will damp the solution for sufficiently large times and hence avoid the recurrence effect. This method has been studied by Knorr [5]. Another method of avoiding recurrence consists in damping selectively the coefficients b_ν when ν is close to n. This corresponds to a smoothing of the distribution function if the ripples in velocity space exceed a certain steepness. The selective damping can be accomplished by adding a term

$$-\epsilon\nu^{2r+1}b_\nu \tag{5}$$

to the right-hand side of Eq. (4) (ϵ is of the order of $n^{-(2r+1)}$, where r is an integer [6]), which is equivalent to adding a collision operator to the right-hand side of Eq. (1):

$$C(v)^{2r+1}f = \epsilon[(\partial/\partial v)(\partial/\partial v) + v)]^{2r+1} f. \tag{6}$$

Knorr and Shoucri [6] proposed the use of a slightly different collision operator, added to the right-hand side of Eq. (1), of the form:

$$(\partial^2/\partial x^2)\, C(v)^{2r+1} f. \tag{7}$$

The term $\partial^2/\partial x^2$, which is added formally in expression (7) results in a diffusion in configuration space which causes the Fourier modes having the highest wave-numbers (hence the lowest recurrence time) to be damped selectively, in addition to the selective damping of the high ν coefficients effected by the operator $C(v)^{2r+1}$.

The operator in expression (7) has been used by Shoucri and Knorr [7] for the Chebyshev representation of the Vlasov equation, but has not been tried previously for the Hermite representation. Situations where the Hermite representation can be more advantageous than the Chebyshev representation have been discussed in [7].

To compare with results previously obtained using the Chebyshev representation, the case of a symmetric two-stream instability has been studied with the initial condition:

$$f(x, v, 0) = (1/(2\pi)^{1/2})\, v^2 \exp(-v^2/2)(1 + A \cos kx) \tag{8}$$

with $A = 0.05$ and $k = 0.5$. Equation (4) with the collision operator given in expression (7) was solved numerically using a leapfrog scheme, initialized by a

Lax–Wendroff step, which was discussed by Knorr [5]. The results are shown in Figs. 1 and 2. Figure 1 gives the magnitude of the first two harmonics $k = 0.5$, 1 as a function of time. In Fig. 2, the total electric energy is plotted linearly in time; as will be observed, it follows the characteristic exponential growth, saturation, and oscillations of the electric field, due to the trapping of the particles. These plots have the general physical features of those reported in [5] and [7]. They have been obtained with 30 polynomials and 16 points in space, i.e., with an amount of information equivalent to 480 "particles." The damping term in expression (7) has been applied with $\epsilon = 5.0/(30)^3$.

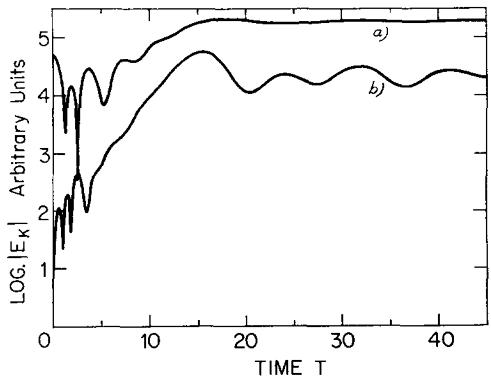

FIG. 1. Plot of the first two harmonics of the electric field (in arbitrary units) for a two-stream instability with the initial condition $f(x, v, 0) = (1/(2\pi)^{1/2}v^2 \exp(-v^2/2)\ (1 + A \cos kx)$ with $A = 0.05$ and $k = 0.5$. The figure gives the evolution in time of the logarithm of the absolute value of the first two modes: (a) $k = 0.5$; (b) $k = 1$.

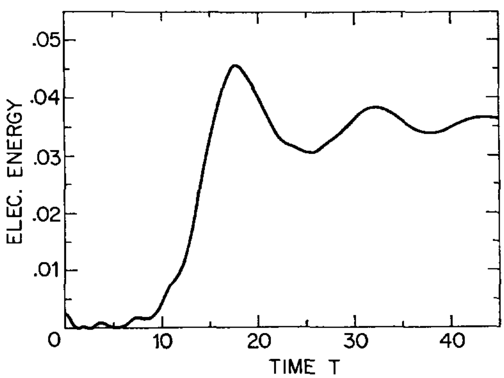

FIG. 2. Plot of the total electric field energy against time.

To further illustrate the applicability of expression (7), Fig. 3 shows the result obtained when studying the evolution of a strongly nonlinear Landau damping. The initial condition is:

$$f(x, v, 0) = (1/(2\pi)^{1/2}) \exp(-v^2/2)(1 + A \cos kx), \tag{9}$$

with $A = 0.5$ and $k = 0.5$. The time evolution of the fundamental mode $k = 0.5$ (curve (a) in Fig. (3)) is very similar to the one reported in [5]: a damping which is much stronger than that obtained when using the linear theory is observed for the initial part of the curve, and a decay of the amplitude by roughly $1\frac{1}{2}$ orders of magnitude occurs. Then the amplitude grows back and settles to a fairly constant

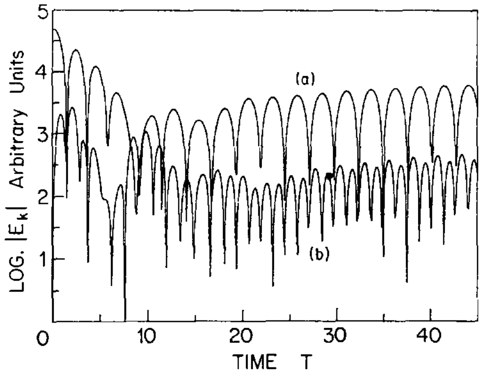

FIG. 3. Plot of the first two harmonics of the electric field (in arbitrary units) for a strongly non-linear Landau damping with the initial condition $f(x, v, 0) = (1/(2\pi)^{1/2} \exp(-v^2/2)(1 + A \cos kx)$ with $A = 0.5$ and $k = 0.5$. The figure gives the evolution in time of the logarithm of the absolute value of the first two modes (a) $k = 0.5$; (b) $k = 1$.

level. The results obtained for the second mode $k = 1$ are represented by curve (b) of Fig. (3). Although, at the beginning, the time evolution follows closely the results reported in [5], the recurrence maximum is observed to occur at a time $t \simeq 10$ rather than $t = 15$ as in [5]; moreover, the amplitude differences between the peaks and the valleys surrounding this maximum are less pronounced than in [5]. Different factors can explain these differences. Most important is the fact that the method used in [5] to damp the recurrence effects did not effect a selective damping of higher k modes (as effected here by the operator given in expression (7)). As a result of this, the higher modes which travel faster than the fundamental mode are less damped than the latter when they recur. Also, the results reported in [5, Fig. (3)] have been calculated with only 8 points in x, while 16 points have been used for the present calculations; this provides a better resolution for higher

k modes. In agreement with the results in [5], no sudden decay is observed after $t = 40$, contrary to what was reported by Nuehrenberg [9].

The results obtained indicate that the scheme performs correctly. Work is in progress to use it in a two-dimensional scheme for the numerical solution of the Vlasov equation [8].

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